The Binomial to the Multinomial

A binomial is often a useful distribution when modelling the counts of independent and identically distributed (i.i.d) bernoulli trials. But how can we model trials with more than two possible outcomes?

The content mentors were split on a very important issue: how to eat corn. Some of them ate the corn horizontally (think typewriter-style), while others ate the corn in spirals from left to right. To get to the bottom of who was right, they decided to hold a survey of the CS70 student population. But because they were too lazy to create a google form, they replaced each student with a biased coin with probability p of landing heads and flipped it. They then made the arbitrary rule that heads meant the student ate the corn horizontally and tails meant the student ate the corn in spirals.

(a) Suppose there are n students, find the probability that k of them ate the corn horizontally.

Solution: Let X denote the number of students that eat corn horizontally. This is just the binomial distribution $\mathbb{P}[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$

(b) A rumour was recently spread that some people ate corn in a double-helix pattern (like a DNA strand). In the spirit of the scientific method, the content mentors decided they should add this options to their test. But since they couldn't find an object with three faces, they had to code up a simulation in python with probability p of the student eating the corn horizontally, probability q of the student eating the corn in spirals, and probability r of the student eating the corn in a double-helix pattern such that $p + q + r = 1$. Find the probability that k student ate the corn in a double-helix pattern.

Solution: Let Z denote the number of students that eat corn in double helices. This is once again a binomial distribution, but this time the probability of success is r and the probability of failure is $1 - r = p + q$. Hence, $\mathbb{P}[Z = k] = \binom{n}{k} r^k (p+q)^{n-k}$

(c) But the content mentors are hungry for data, so they want to know something more specific: what is the probability that k students ate the corn in a double-helix pattern **and** m student ate the corn in spirals. Simplify your expression.

Solution: let Z have the same definition as last part and Y denote the number of students that each corn in spirals. Then, we care to find $\mathbb{P}[Z = k \cap Y = m] = \mathbb{P}[Z = k]\mathbb{P}[Y = m | Z = k]$. The first term is known from part b, but the second term is also binomial since now there are only $n - k$ unknowns. Since each unknown has to be either a spiral or a a horizontal, the probability of success is $\frac{q}{p+q}$. Hence $\mathbb{P}[Y = m | Z = k] = \binom{n-k}{m} (\frac{q}{p+q})^m (\frac{p}{p+q})^{n-k-m}$ and

$$\mathbb{P}[Z = k \cap Y = m] = \binom{n}{k}\binom{n-k}{m} r^k (p+q)^{n-k} \frac{q^m p^{n-k-m}}{(p+q)^{n-k}} \tag{1}$$

$$= \frac{n!}{k!m!(n-k-m)!} r^k q^m p^{n-k-m} \tag{2}$$

This is the formula for a multinomial $M(n, r, q)$!