Riemann's Pontiac

Your friend Riemann drives a 1994 Pontiac Firebird, an old car and that he hasn't taken care of it all too well. At every second in $\{t_1, \ldots, t_n\}$ with probability p he accelerates at $1\ m/s^2$ and with probability $1-p$ his engine sputters and he stops immediately. Find the expected time it takes him to reach a velocity of $n\ m/s$.

Solution: To keep things simple (we're not in a physics class!), we're going to remove the units and look only at the numbers. Fortunately, everything is in meters and seconds so we won't have to worry about any conversions.

If we draw the markov chain with each state representing some velocity from 0 to n, then we see that the transition probability for any velocity v_k to v_{k+1} is p and the transition from v_k to v_0 occurs with probability $1-p$.

Hence, we get the hitting time equation:

$$\beta(v_n) = 0 \tag{1}$$
$$\beta(v_k) = 1 + p\beta(v_{k+1}) + (1-p)\beta(v_0)\ \forall 0 \leq k \leq n \tag{2}$$

If we recurse backwards from v_n we get a rather ugly set of equations. For the first two steps,

$$\beta(v_{n-1}) = 1 + (1-p)\beta(v_0) \tag{3}$$
$$\beta(v_{n-2}) = 1 + p(1 + (1-p)\beta(v_0)) + (1-p)\beta(v_0) \tag{4}$$

These equations are quite nasty to deal with, so while hitting time would give us the correct answer, it would be too cumbersome to work out. Instead, let's use the random variable W_k to denote the the number of timesteps before reaching a velocity of k. Also denote the event where we accelerate on the next timestep with A. Then,

$$\mathbb{E}(W_k|W_{k-1}) = \mathbb{E}(W_k|W_{k-1}, A)\mathbb{P}(A) + \mathbb{E}(W_k|W_{k-1}, \bar{A})\mathbb{P}(\bar{A}) \tag{5}$$
$$= p(1 + \mathbb{E}(W_{k-1})) + (1-p)(1 + \mathbb{E}(W_{k-1}) + \mathbb{E}(W_k)) \tag{6}$$
$$= 1 + \mathbb{E}(W_{k-1}) + (1-p)\mathbb{E}(W_k) \tag{7}$$

Applying iterated expectation,

$$\mathbb{E}(W_k) = 1 + \mathbb{E}(\mathbb{E}(W_{k-1})) + (1-p)\mathbb{E}(\mathbb{E}(W_k)) \tag{8}$$
$$\mathbb{E}(W_k) = 1 + \mathbb{E}(W_{k-1}) + (1-p)\mathbb{E}(W_k) \tag{9}$$

Hence,

$$\mathbb{E}(W_k) = \frac{1}{p} + \frac{1}{p}\mathbb{E}(W_{k-1}) \tag{10}$$

From here we can make a clever realization. Let $g(k) = \mathbb{E}(W_k) + \frac{1}{1-p}$. Then,

$$g(k) = \frac{1}{p} + \frac{1}{p}\mathbb{E}(W_{k-1}) + \frac{1}{1-p} \tag{11}$$
$$= \frac{1}{p}(\mathbb{E}(W_{k-1}) + \frac{1}{1-p}) \tag{12}$$
$$= \frac{1}{p}g(k-1) \tag{13}$$

Now that we have transformed this problem a geometric sequence, we have a closed form solution to the reccurence:

$$g(k) = \frac{1}{p^k}g(0) \tag{14}$$
$$= \frac{1}{p^k(1-p)} \tag{15}$$

Now, we must transform back to $\mathbb{E}(W_k)$.

$$\mathbb{E}(W_{k-1}) = g(k) - \frac{1}{1-p} \tag{16}$$

$$= \frac{1}{p^k(1-p)} - \frac{1}{1-p} \tag{17}$$

If you prefer, you can also try expanding a few terms, guessing a formula and using induction to prove it holds (the nice thing about recurrence relations are that they are pretty easy to prove with induction).

Thus, the expected time it takes to reach a velocity of nm/s is $\frac{1}{p^n(1-p)} - \frac{1}{1-p}$ seconds.