One D -> Two D

Recall from long, long ago, the binary truth operators OR, AND, XOR, and so forth. In this problem we analyze these operators as probability distributions.

(a) Suppose X, Y are discrete random variables taking on values in $\{0, 1, 2, 3\}$ and let $\mathbb{P}(X = x, Y = y) = c(x \oplus y)$. $\oplus$ is the XOR operator, and can be expressed in terms of elementary logic operators as $P \oplus Q = (P \vee Q) \wedge \neg(P \wedge Q)$.

Since X and Y are not constrained to 0 or 1, in this case $\oplus$ is applied bitwise (e.g. $3 \oplus 2 = 11_2 \oplus 10_2 = 01_2 = 1$).

Find c, and express the joint distribution of X and Y with a probability table, and find the marginal distributions for X and Y.

Solution: First, we write the table in terms of c: (Notice it doesn't matter what axis is X or Y since the distribution is symmetric).

	0	1	2	3
0	0	c	2c	3c
1	c	0	3c	2c
2	2c	3c	0	c
3	3c	2c	c	0

The sum over all values for x and y must be 1, so for this to be a valid probability distribution,

$$\sum_{x=0}^{3} \sum_{y=0}^{3} \mathbb{P}(X = x, Y = y) = 24c = 1 \tag{1}$$

so $c = \frac{1}{24}$. Now, notice that each row and each column sum to $\frac{1}{4}$, hence the marginal distribution is

$$P(X = x) = P(Y = y) = \frac{1}{4} \tag{2}$$

(b) Using the same definitions from part (a), find the covariance and the correlation of X, Y [5cm]

Solution: Using the definition of covariance,

$$Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \tag{3}$$

X and Y are uniform and identically distributed so $\mathbb{E}[X] = \mathbb{E}[Y] = \frac{3}{2}$

To find $\mathbb{E}[XY]$, we sum over all values for X, Y and find the weighted average (if this formula doesn't make sense to you, you can also define a new variable $Z = XY$, and take the expectation of that):

$$\mathbb{E}[XY] = \sum_{x=0}^{3} \sum_{y=0}^{3} xy\mathbb{P}(X = x, Y = y) \tag{4}$$

$$= 2 \cdot \frac{3}{24} + 3 \cdot \frac{2}{24} + 6 \cdot \frac{1}{24} + 2 \cdot \frac{3}{24} + 3 \cdot \frac{2}{24} + 6 \cdot \frac{1}{24} \tag{5}$$

$$= \frac{3}{2} \tag{6}$$

Hence, we have

$$Cov(X,Y) = \frac{3}{2} - (\frac{3}{2})^2 \tag{7}$$
$$= -\frac{3}{4} \tag{8}$$
$$\tag{9}$$

$\sigma_X = \sigma_Y = \sqrt{Var(X)} = \sqrt{\frac{4^2-1}{12}} = \sqrt{\frac{5}{4}}$ using the formula for variance of a uniform distribution. Hence, using the formula for correlation,

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = -\frac{3}{5} \tag{10}$$

This makes sense since if you plot the graph, you will notice a negative correlation.

(c) Find $\mathbb{E}[X+Y]$ and $Var[X+Y]$

Solution: By linearity of expectation, $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 3$.

By bilinearity of covariance,

$$Var(X+Y) = Cov(X+Y, X+Y) \tag{11}$$
$$= Cov(X,X) + 2Cov(X,Y) + Cov(Y,Y) \tag{12}$$
$$= Var(X) + 2Cov(X,Y) + Var(Y) \tag{13}$$
$$= \frac{5}{4} - \frac{6}{4} + \frac{5}{4} \tag{14}$$
$$= 1 \tag{15}$$