Consider a little game. Alice and Bob arrange 8 coins in a circle with alternating faces up (i.e. Heads, Tails, Heads, Tails, ...), and they each take turns flipping coins according to the following rules:

Alice chooses any coin with heads facing up and flips it over. Then, she sets the coin adjacent to it in the clockwise direction to heads.
Bob chooses any coin with tails facing up and flips it over. Then, he sets the coin two spots away (index $i + 2$) in the clockwise direction to tails.

Assume Alice always plays the first move.

(a) How many distinct ways are there for Alice and Bob to play the first two moves (hint: how many unique tuples (i, j) exist where i is the index of the first coin being flipped and j is the index of the second).

> **Solution: 16**
>
> There are initially 4 coins with heads, meaning i can take on 4 possible values. Now, no matter what coin Alice picks, the coin adjacent to it in the clockwise direction is always set to tails. Hence, Alice is flipping one coin from heads to tails and one coin from tails to heads, so the total number of Heads and Tails after her turn are 4 each.
>
> Now, on Bob's turn, j can take on 4 possible values, since there are 4 tails. Hence, by the First Rule of Counting, there are a total of $4 \cdot 4 = 16$ ways of selecting the indices for Alice and Bob's first two moves.

(b) Suppose the game ends when there are no legal moves for the next player. Prove that no matter what Alice and Bob play, the game will never end.

> **Solution:** If there are no legal moves for a given player, then all the coins must have the same face (that is, either all heads or all tails). We prove the contrapositive of this statement: if at every turn, there is at least one head and at least one tail, then there will always be a legal move.
>
> First, we notice that at every turn, the number of heads or tails increases by 0 or 1. Hence, before there are no heads or no tail, there must be a turn where there is one head or one tail.
>
> **Case 1**: If there is one head, then either it is Bob's turn, and the number of heads either stays the same or increases, or it is Alice's turn and she flips it to a tail. However, Alice must set the coin adjacent to it to a head, so the number of heads remains 1.
>
> **Case 2**: If there is one tail, then either it is Alice's turn, and the number of tails either stays the same and increases, or it is Bob's turn and he flips the tail to a head. Now, Bob would have to set the coin two places clockwise to a tail, meaning that the number of tails remains 1.
>
> Since there is always at least one tail and one head, the game can never end.

(c) Use parts (a) and (b) to provide an upper bound to the number of distinct ways for Alice and Bob to play the first k moves. Assume k is even.
Hint: to find an upper bound, think about the maximum number of choices Alice/Bob can make at each step.

> **Solution:** $4^2 \cdot 7^{k-2}$
>
> Using part (a), the first two moves can be played in 4^2 ways. From part (b), we determined that the game never ends, meaning there is always at least one head or tail on the board. Hence, Alice or Bob can choose from at most 7 coins on

every subsequent turn. Using the First Rule of Counting, thus, we provide an upper bound of $4^2 \cdot 7^{k-2}$

(d) It turns out that a little optimization can lead us to a pretty tight upper bound. Prove that you can bound the number of distinct ways for Alice and Bob to play the first k moves by $4^2 \cdot 20^{\frac{k}{2}-1}$

Solution: Since at every turn, we notice that if Alice has h heads to flip from, then after her turn, the number of tails for Bob to pick from is either $8-h$ or $8-h+1$. Hence, the number of moves in any pair of turns $(k, k+1)$ is at most $h(8-h+1)$ by the First Rule of Counting. This function is maximized when $h=4$ or $h=5$ (provable by taking the first and second derivative or simply by realizing that this is a quadratic polynomial and finding the vertex), giving us an upper bound of 20 moves for any pair of turns. Since we proved that the game never ends, there is no line of play that terminates within the first k moves. Hence, we can once again use the First Rule of Counting to upper bound the number of moves in turns $3 \ldots k$ by $20^{\frac{k-2}{2}}$. Since there are 16 ways of arranging the first two moves, we apply the first rule of counting once again, giving us an upper bound of $4^2 \cdot 20^{\frac{k}{2}-1}$

(e) How many distinct arrangements are there for the coins after 2 moves have been played? Assume that any arrangement of coins that can be reached by rotating all the coins by $1 \leq i \leq 7$ indices is equivalent.

Solution: 3

Suppose the indices of the heads before Alice makes her move are 1, 3, 5, and 7. Now, after she makes her move, the indices of the heads could be:

2,3,5,7
1,4,5,7
1,3,6,7
1,3,5,8

We also know that the indices are unique modulo 8 (rotating the entire circle by 8 indices leaves all indices at their original positions). Hence, rotating the second configuration by 6 indices clockwise, the third configuration by 4 and the fourth configuration by 2 all modulo 8 gives us:

$(2,3,5,7) \pmod 8$
$(7,2,3,5) \pmod 8$
$(5,7,2,3) \pmod 8$
$(3,5,7,2) \pmod 8$

These are all equivalent arrangements though, since all the heads are at the same indices.

Now, we can without loss of generality consider the arrangement $THHTHTHT$. After Bob takes his turn, the resulting arrangements are

$HHTTHTHT$
$THHHHTHT$
$THHTHHHT$
$TTHTHTHH$

Now, we notice that the fourth arrangement can be rotated by 2 indices to reach the first arrangement. Hence, there are only 3 unique arrangements after 2 turns.