Suppose two integers a and b are drawn uniformly from $[-n \mathinner{..} n]$, that is $a, b \in \mathbb{Z}$ and $-n \leq a, b \leq n$.

(a) Define a probability space for (a, b). Does each sample point occur with uniform probability?

Solution: A probability space is defined by a sample space and probabilities for each sample point. The sample space is defined as

$$\Omega = \{(i, j) : -n \leq i, j \leq n\}$$

Clearly $\mathbb{P}[(i, j)] = \mathbb{P}[(k, l)]$ by uniform symmetry, so $\forall \omega \in \Omega$,

$$\mathbb{P}[\omega] = \frac{1}{|\Omega|} = \frac{1}{(2n+1)^2} \tag{1}$$

(b) Find the probability that $\max\{0, a\} = \min\{0, b\}$.

Solution: The problem is true if and only if $a \leq 0$ and $b \geq 0$. There are $n+1$ such values for a and $n+1$ such values for b, meaning there are $(n+1)^2$ sample points. Given each sample point occurs with equal probability, we simply divide the number of sample points by the cardinality of the sample space:

$$\frac{(n+1)^2}{(2n+1)^2}$$

(c) Find the probability that $|a - b| \leq k$. You may assume $k < \frac{n}{2}$.

Solution: We begin by finding a formula for arbitrary n.
If $a < -n + k$, then $a = -n + i$ where $i \in [0 \mathinner{..} k-1]$ then, b can be in $[-n \mathinner{..} -n+k+i]$, meaning there are $k + i + 1$ satisfying sample points for b.

If $-n + k \leq a \leq n - k$, then b can be in $[a - k \mathinner{..} a + k]$, meaning there are $2k + 1$ satisfying sample points for b.

If $a > n - k$, the case is symmetric to the first case and there are again $k + i + 1$ satisfying sample points for b.

Now, we find the number of sample points for (a, b) by summing over all possible values for a:

$$
\begin{aligned}
|A| &= 2\Big(\sum_{i=0}^{k-1} k + i + 1\Big) + \Big(\sum_{i=1}^{(n-k)-(-n+k)+1} 2k + 1\Big) && (2) \\
&= 2\big(k^2 + \frac{k(k-1)}{2} + k\big) + (2n - 2k + 1)(2k + 1) && (3) \\
&= 2k(k+1) + k(k-1) + (2n - 2k + 1)(2k + 1) && (4) \\
&= -k^2 + k + 4nk + 2n + 1 && (5)
\end{aligned}
$$

From (1) to (2), we use the arithmetic sum formula. Since each sample point occurs with equal probability,

$$\mathbb{P}(|A|) = \frac{-k^2 + k + 4nk + 2n + 1}{(2n+1)^2} \tag{6}$$

(d) Suppose we choose two closed intervals $u = [a \mathinner{..} b]$, $v = [c \mathinner{..} d]$ uniformly at random from $[-n \mathinner{..} n]$. What is the probability that u is enveloped by v, meaning that $u \subset v$ and $c < a \leq b < d$. What happens to this probability as n approaches ∞?

Solution: First, we define our sample space as the set of all pairs of intervals, so that the probability of any sample point is equal.

Notice that any interval is uniquely defined by its starting and ending index. Hence, we place bars between elements, before the first element, and after the last element (e.g. $|-n|-n+1|\ldots|n-1|n|$) and the number of intervals is the number of ways to choose 2 bars, which is the same as choosing 2 distinct elements from $2n+2$.

$$\binom{2n+2}{2} \tag{7}$$

Hence, the size of the sample space is $\binom{2n+2}{2}^2$.

Now, there is always exactly one arrangement of (a, b, c, d), (e, f, g, h) such that $e < f \leq g < h$. Once again, this problem can be modeled as selecting 4 bars from the bars between elements, giving us

$$|A| = \binom{2n+2}{4} \tag{8}$$

Dividing $|A|$ by $|\Omega|$ gives

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{\binom{2n+2}{4}}{\binom{2n+2}{2}^2} = \frac{1}{6}\frac{2n!^2}{(2n-2)!(2n+2)!} = \frac{1}{6}\frac{2n(2n-1)}{(2n+2)(2n+1)} \tag{9}$$

As $n \longrightarrow \infty$, the probability converges to $\frac{1}{6}$

(e) Suppose Ezekiel places m rectangles on an nxn grid, each chosen uniformly at random and independent from all others. Lower bound the probability that no rectangle is enveloped by another rectangle (that is, without any edges touching, the area of one rectangle is located entirely inside the area of another rectangle). Assuming n is sufficiently large, express this bound as a function of m

Solution: First, we notice that for no rectangle to be enveloped by another rectangle, there must be no pairs (r_i, r_j) of rectangles r_i and r_j such that r_i envelopes r_j.

Hence, let E_{ij} be the event where r_i envelopes r_j and N be the event wherein no rectangle is enveloped by another. Then,

$$\mathbb{P}(N) = \mathbb{P}(\bigcap_{1 \leq i \neq j \leq m} \bar{E}_{ij}) \tag{10}$$

$$= 1 - \mathbb{P}(\bigcup_{1 \leq i \neq j \leq m} E_{ij}) \tag{11}$$

Now, we can apply the union bound

$$\mathbb{P}(N) \geq 1 - \sum_{1 \leq i \neq j \leq m} \mathbb{P}(E_{ij}) \tag{12}$$

$$= 1 - m(m-1)\mathbb{P}(E_{ij}) \tag{13}$$

Here we make use of symmetry since $\mathbb{P}(E_{ij})$ should be the same regardless of the rectangles we choose.

Now, we calculate $\mathbb{P}(E_{ij})$. Notice first that any rectangle can be uniquely represented by a left edge, a right edge, a top edge, and a bottom edge. Also, notice that the location of the left edge and right edge are entirely independent of the locations of the top edge and bottom edge, since each rectangle appears with uniform probability.

Now, we can represent the index of the left and right edges as x_l and x_r, and the locations of the top and bottom edges as y_t and y_b. Thus, the sufficient and necessary conditions for E_{ij} to be satisfied are:

1) $x_{li} < x_{lj} < x_{rj} < x_{ri}$
2) $y_{bi} < y_{bj} < y_{tj} < y_{ti}$

Notice that both of these subproblems are the interval enveloping problem discussed in 1d). Only in this case, we have an interval length of $n - 1$ instead of $2n + 1$ (while the grid has length n, we require that rectangles have length at least 2, whereas intervals in 1d could have length 1. Suppose X_s is the event where the first condition is satisfied and Y_s is the condition where the second condition is satisfied. Then, by independence

$$\mathbb{P}(E_{ij}) = \mathbb{P}(X_s \cap Y_s) = \mathbb{P}(X_s)\mathbb{P}(Y_s) = \left(\frac{1}{6}\frac{(n-2)(n-3)}{n(n-1)}\right)^2 \tag{14}$$

$$= \frac{1}{36}\frac{(n-2)^2(n-3)^2}{n^2(n-1)^2} \tag{15}$$

Plugging this value back into (13) gives us the lower bound

$$\mathbb{P}(N) \geq 1 - \frac{m(m-1)}{36}\frac{(n-2)^2(n-3)^2}{n^2(n-1)^2} \tag{16}$$

The right term converges to 1, as $n \to \infty$ (proof isn't included here, but notice that the leading term of the expanded polynomial fraction has a coefficient of 1 in both the numerator and denominator).

Hence, we get rather a nice lower bound of

$$\mathbb{P}(N) \geq 1 - \frac{m(m-1)}{36} \tag{17}$$